Exponential Derivatives

$$f(x) = a^x \quad (a > 0) \quad \Rightarrow \quad f'(x) = ?$$

From Definition : $\displaystyle \frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \left(\frac{a^h - 1}{h} \right)$$

$$= \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right) a^x$$

This is same fixed number and does not depend on x

Problem : $\left.\begin{array}{l} \lim_{h \to 0} a^h - 1 = a^0 - 1 = 1 - 1 = 0 \\ \lim_{h \to 0} h = 0 \end{array}\right\} \Rightarrow$ Unclear Quotient

Unfortunately no obvious way to simplify $\dfrac{a^h - 1}{h}$. We'll have to rely on table.

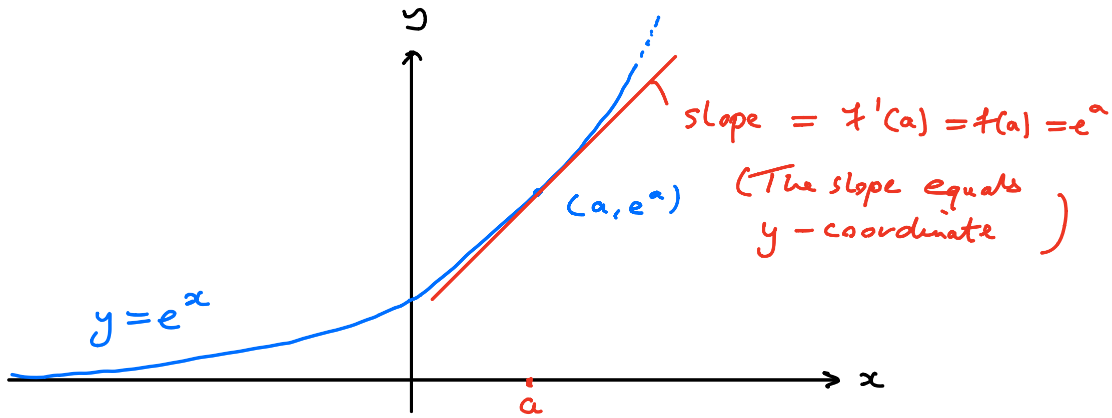
$(e = 2.718\ldots)$

h	$a=2$: $\frac{2^h - 1}{h}$	$a=3$: $\frac{3^h - 1}{h}$	$a=2.5$: $\frac{(2.5)^h - 1}{h}$	$a=e$: $\frac{e^h - 1}{h}$
0.1	0.7177	1.1612	0.9596	1.0517
0.01	0.6959	1.1047	0.9205	1.0056
0.001	0.6934	1.099	0.9167	1.0005
−0.001	0.6929	1.098	0.9159	0.995
−0.01	0.6907	1.0926	0.9121	0.9956
−0.1	0.6697	1.0404	0.8756	0.9516

Conclusion : $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$\Rightarrow \boxed{\frac{d}{dx}(e^x) = e^x}$$

This is the reason e is so special.



Q/ : $y = e^{g(x)} \Rightarrow \frac{dy}{dx} = ?$

Let's use chain rule : $y = e^u$, $u = g(x)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{d(e^u)}{du} \cdot \frac{d\,g(x)}{dx}$$
$$= e^u \cdot g'(x) = e^{g(x)} \cdot g'(x)$$

Conclusion : $\boxed{\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)}$

Example : 1/ $f(x) = e^{(x^2)} \Rightarrow f'(x) = ?$

(Warning: $f(x) \neq (e^x)^2 = e^{2x}$)

$f(x) = e^{g(x)}$ where $g(x) = x^2$. $\Rightarrow g'(x) = 2x$

$\Rightarrow f'(x) = e^{x^2} \cdot 2x$

2/ $y = \dfrac{2x+1}{e^{(3x+1)}} \Rightarrow \dfrac{dy}{dx} = ?$

$y = \dfrac{u}{v}$ where $u = 2x+1$ and $v = e^{3x+1}$

Quotient Rule.

$\Rightarrow \dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$

$u' = \dfrac{du}{dx} = 2$

$v' = \dfrac{dv}{dx} = \dfrac{d}{dx}\left(e^{(3x+1)}\right) = e^{(3x+1)} \cdot \dfrac{d(3x+1)}{dx}$

$= e^{(3x+1)} \cdot 3$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2 \cdot e^{(3x+1)} - (2x+1) \cdot e^{(3x+1)} \cdot 3}{\left(e^{(3x+1)}\right)^2}$

Q/ : What is $\dfrac{d}{dx}(a^x)$ when $a \neq e$?

Recall :

$a = e^{\ln(a)} \Rightarrow a^x = \left(e^{\ln(a)}\right)^x = e^{(\ln(a)x)}$

just a constant

$\Rightarrow a^x = e^{g(x)}$ where $g(x) = \ln(a)\, x$

$\Rightarrow \dfrac{d}{dx}(a^x) = e^{g(x)} g'(x) = e^{\ln(a)x} \ln(a) = a^x \ln(a)$

Conclusion :

$$\boxed{\frac{d}{dx}(a^x) = \ln(a)\, a^x}$$

Remark : Going back to initial observations :

$$\frac{d}{dx}(a^x) = \left(\lim_{h\to 0} \frac{a^h - 1}{h}\right) a^x = \ln(a)\, a^x$$

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln(a)$$

E.g. $\ln(2) = 0.693147$
$\ln(3) = 1.09861$
$\ln(2.5) = 0.91629$
$\ln(e) = 1$

These are the limits in our above table.

Examples 1) $y = \dfrac{1}{2^x} \Rightarrow \dfrac{dy}{dx} = ?$

$$y = \frac{1}{2^x} = 2^{-x} = (2^{-1})^x = \left(\tfrac{1}{2}\right)^x$$

$$\Rightarrow \frac{dy}{dx} = \ln\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2}\right)^x = -\ln(2)\, 2^{-x}$$

2) $y = \dfrac{5}{1 - 2^t} \Rightarrow \dfrac{dy}{dt} = ?$

Method 1

$$y = \frac{5}{1-2^t} = 5\cdot(1-2^t)^{-1}$$

$$\Rightarrow \quad y = 5 \cdot u^{-1}, \quad u = 1 - 2^t$$

Chain Rule

$$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

$$= \left(5 \cdot (-1)\, u^{-2}\right) \cdot \left(-\ln(2)\, 2^t\right)$$

$$= \left(-5\,(1-2^t)^{-2}\right) \cdot \left(-\ln(2)\, 2^t\right)$$

<u>Method 2</u>

$$y = \frac{u}{v} \quad \text{where} \quad u = 5$$

$$v = 1 - 2^t$$

$$\Rightarrow \quad \frac{du}{dt} = 0, \quad \frac{dv}{dt} = -\ln(2)\, 2^t$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{\frac{du}{dt} \cdot v - u \cdot \frac{dv}{dt}}{v^2}$$

$$= \frac{0 \cdot (1-2^t) - (5) \cdot (-\ln(2)\, 2^t)}{(1-2^t)^2}$$

$$= (-5)\,(1-2^t)^{-2}\,(-\ln(2)\, 2^t)$$